Optimizing Lateral Boundary Conditions at Staircase-shaped Coastlines: Variational Approach.

Christine & Eugene Kazantsev

Unversity Grenoble Alpes & INRIA
FRANCE

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Институт Вычислительной Математики им. Г.И. Марчука
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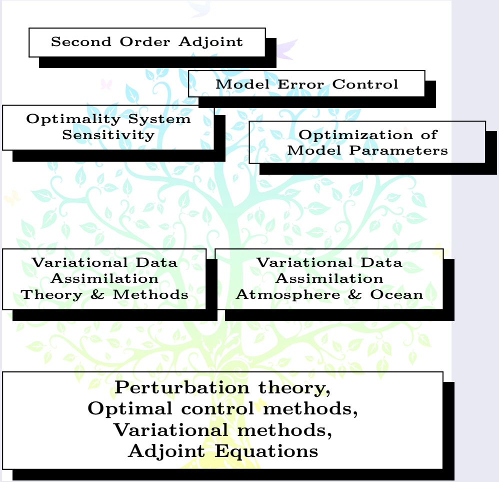
Layout:

2000-...

1980-2010

1960-1985



Development

Layout:

2000-...

- Second Order Adjoint
- Model Error Control
- Optimality System Sensitivity
- Optimization of Model Parameters

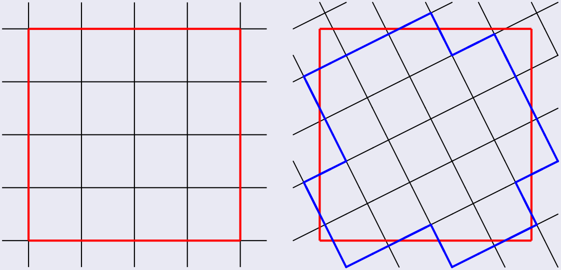
1980-2010

- Variational Data Assimilation Theory & Methods
- Variational Data Assimilation Atmosphere & Ocean

1960-1985

- Perturbation theory, Optimal control methods, Variational methods, Adjoint Equations

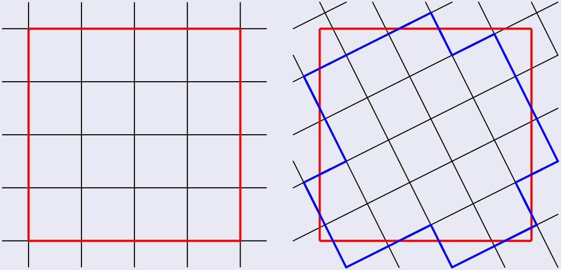
Staircase-shaped boundary



When the grid is not aligned with the model boundary, boundary conditions become difficult to prescribe.

Boundary conditions at rigid boundaries

Staircase-shaped boundary



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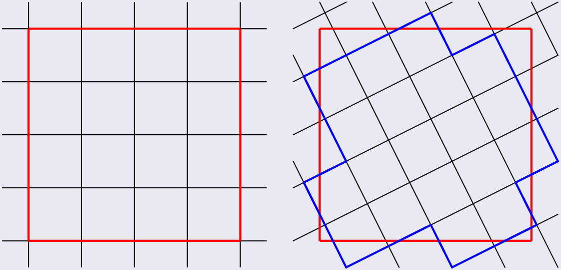
- Finite elements? Expensive...
-
-

Staircase-shaped boundary



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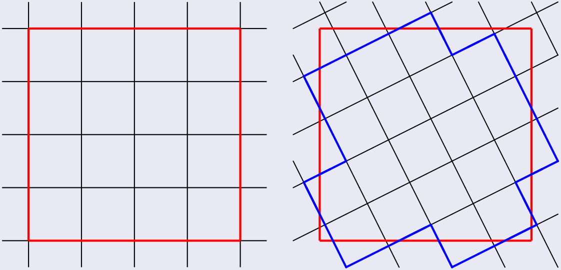
- Finite elements? Expensive...
- Better interpolation in frames of FD? May be instable...
-

Staircase-shaped boundary



When the grid is not aligned with the model boundary, boundary conditions become difficult to prescribe.

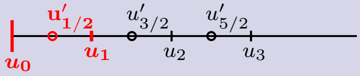
- Finite elements? Expensive...
- Better interpolation in frames of FD? May be instable...
- Variational methods ? We can try...

Discretized derivatives

Particular discretisation for derivatives near the boundary

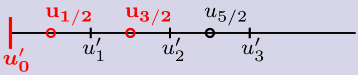
Boundary conditions are introduced into the model by a particular discretization of operators near the boundary.
To avoid instabilities, we control both BC and their approximation.

Integer nodes



$$\text{if } u_0 = 0 \longrightarrow \left.\frac{\partial u}{\partial x}\right|_{1/2} = \frac{u_1}{h}$$

Half-integer nodes



$$\text{if } u_0 = 0 \longrightarrow \left.\frac{\partial u}{\partial x}\right|_{0} = \frac{2u_{1/2}}{h}$$

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Particular discretisation for derivatives near the boundary

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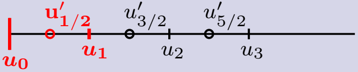
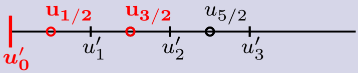


$$\text{if } u_0 = 0 \longrightarrow \left.\frac{\partial u}{\partial x}\right|_{1/2} = \frac{u_1}{h}$$

$$\left.\frac{\partial u}{\partial x}\right|_{1/2} = \frac{\alpha_1 u_0 + \alpha_2 u_1}{h}$$

Half-integer nodes



$$\text{if } u_0 = 0 \longrightarrow \left.\frac{\partial u}{\partial x}\right|_{0} = \frac{2u_{1/2}}{h}$$

$$\left.\frac{\partial u}{\partial x}\right|_{0} = \frac{\alpha_1 u_{1/2} + \alpha_2 u_{3/2}}{h}$$

where α_1 and α_2 are the control coefficients.

Derivatives are allowed to change their properties near the boundaries in order to find the best fit with requirements of the model and data.

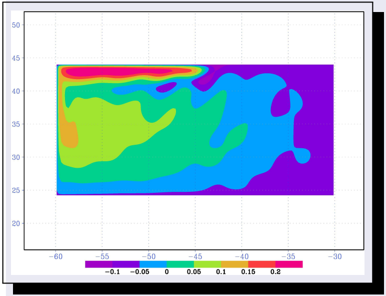
$30^\circ \times 20^\circ$ rectangle with $\frac{1}{4}^\circ$ resolution and 5 z levels.

$120 \times 80 \times 5$ nodes in (x, y, z) coordinates, 64 time steps per day.

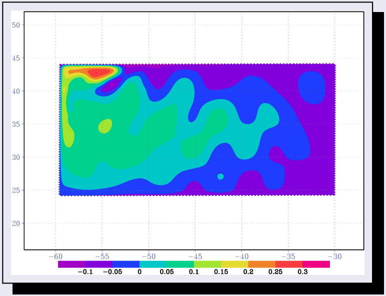
$$
\begin{aligned}
\frac{\partial u}{\partial t} &= v(\omega + f) - \frac{\partial (u^2+v^2)/2}{\partial x} - w\frac{\partial u}{\partial z} - \frac{\partial A_u^h \xi}{\partial x} + \frac{\partial A_u^h \omega}{\partial y} + \\
&+ \frac{\partial}{\partial z} A_u^z \frac{\partial u}{\partial z} + g\int_0^z \frac{\partial \rho(x,y,\zeta)}{\partial x} d\zeta + g\frac{\partial(\eta + T_c\phi)}{\partial x} \\
\xi &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \qquad \omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \quad \text{Divergence, Vorticity} \\
w &= \int_H^z \xi(x,y,\zeta)d\zeta; \quad w(x,y,H) = 0 \quad \text{Vertical velocity} \\
A^z &= 1.2\times 10^{-4}\frac{m^2}{s}; \; A^h = 200\frac{m^2}{s}, \quad f = \frac{4\pi}{86400}\sin(lat) \\
\left.\frac{\partial u}{\partial z}\right|_{w_0} &= \frac{-0.1\frac{N}{m^2}\cos\left(B\pi * \frac{lat - 24^\circ}{44^\circ - 24^\circ}\right)}{hz_1\rho_0} \qquad \begin{array}{ll} B = 2, & \text{Double gyre} \\ B = 1, & \text{Single gyre} \end{array}
\end{aligned}
$$

$$(u_\perp, \omega)_{\text{Lateral Boundary}} = 0 \text{ (Impermeability and Free-Slip conditions)}$$

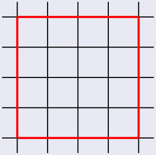
Aligned grid and rotated grid

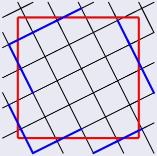


SSH on the aligned grid



SSH on the 45° rotated grid.





$30^\circ \times 20^\circ$ rectangle with $\frac{1}{4}^\circ$ resolution and 5 z levels.

$120 \times 80 \times 5$ nodes in (x, y, z) coordinates, 64 time steps per day.

$$\frac{\partial u}{\partial t} = v(\omega + f) - \frac{\partial (u^2+v^2)/2}{\partial x} - w\frac{\partial u}{\partial z} - \frac{\partial A_u^h \xi}{\partial x} + \frac{\partial A_u^h \omega}{\partial y} +$$

$$+ \frac{\partial}{\partial z} A_u^z \frac{\partial u}{\partial z} + g\int_0^z \frac{\partial \rho(x,y,\zeta)}{\partial x} d\zeta + g\frac{\partial(\eta + T_c\phi)}{\partial x}$$

$$\xi = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \qquad \omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \quad \text{Divergence, Vorticity}$$

$$w = \int_H^z \xi(x,y,\zeta)d\zeta; \quad w(x,y,H) = 0 \quad \text{Vertical velocity}$$

$$A^z = 1.2 \times 10^{-4}\frac{m^2}{s}; \; A^h = 200\frac{m^2}{s}, \quad f = \frac{4\pi}{86400}\sin(lat)$$

$$\left.\frac{\partial u}{\partial z}\right|_{w_0} = \frac{-0.1\frac{N}{m^2}\cos\left(B\pi * \frac{lat - 24^\circ}{44^\circ - 24^\circ}\right)}{hz_1\rho_0} \qquad \begin{array}{ll} B=2, & \text{Double gyre} \\ B=1, & \text{Single gyre} \end{array}$$

$$(u_\perp, \omega)_{\text{Lateral Boundary}} = 0 \text{ (Impermeability and Free-Slip conditions)}$$

$$\left.\frac{\partial v}{\partial x}\right|_{\omega_\circ} = \frac{\alpha_1 v_{1/2} + \alpha_2 v_{3/2}}{h}$$

$$\left.\frac{\partial u}{\partial y}\right|_{\omega_\circ} = \frac{\alpha_1 u_{1/2} + \alpha_2 u_{3/2}}{h}$$

$$\left.\frac{\partial v}{\partial x}\right|_{\omega_\circ} = \frac{\alpha_1 v_{-1/2} + \alpha_2 v_{1/2}}{h}$$

$$\left.\frac{\partial u}{\partial y}\right|_{\omega_\circ} = \frac{\alpha_1 u_{-1/2} + \alpha_2 u_{1/2}}{h}$$

$$\left.\frac{\partial \omega}{\partial x}\right|_{v} = \frac{\alpha_1 \omega_0 + \alpha_2 \omega_1}{h}$$

$$\left.\frac{\partial \omega}{\partial y}\right|_{u} = \frac{\alpha_1 \omega_0 + \alpha_2 \omega_1}{h}$$

$$\alpha_1 = \alpha_2 = 0$$

$$\alpha_1 = -1, \quad \alpha_2 = 1$$

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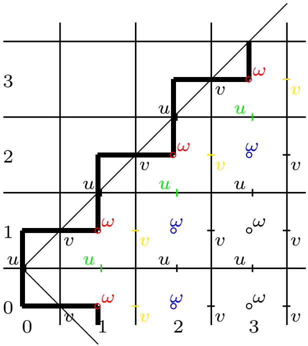


Рис.: 45° rotated grid

The model: $x(t) = \mathcal{M}_{0,t}(x(0), \alpha)$ with $x = (u, v, T, S, ssh)^T$

Cost function J

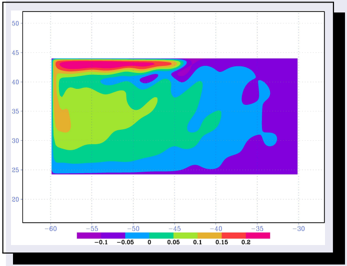
$$
\begin{aligned}
J &= 10^{-4}(\|x(0) - x_{bgr}\|^2 + \|\alpha - \alpha_{bgr}\|^2) + \\
&+ \int_{t=0}^{T} t \iint (u - u_{\text{ref}})^2 + (v - v_{\text{ref}})^2 + (ssh - ssh_{\text{ref}})^2 dxdydt
\end{aligned}
$$

Layout:

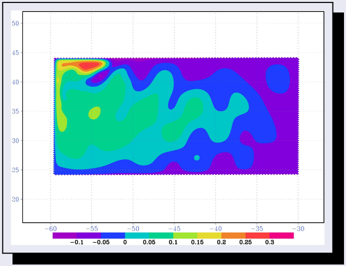
- Joint control of the initial point $x(0)$ (interpolation errors) and the set of α;
- Artificially generated data by the same model on the aligned grid;
- Data Assimilation over the 50 days window;
- Analysis of the solution on the 8 years interval.

- Minimization is performed by M1QN3 (JC Gilbert, C.Lemarechal);
- Adjoint is generated by Tapenade (Ecuador team, INRIA).

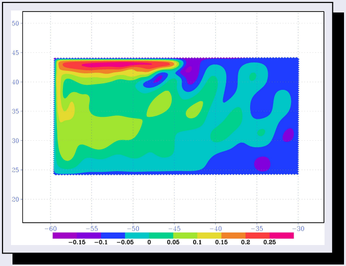
Reference, Optimal and Conventional BC 800 days later



Reference SSH

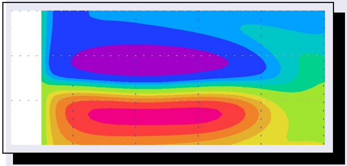


Rotated grid conventional BC SSH

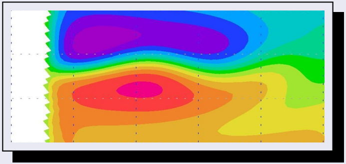


Rotated grid Optimal BC SSH

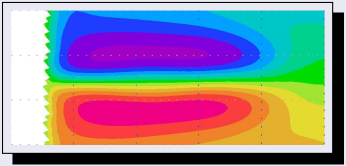
Reference, Rotated, Optimal SSH 10 years average.



Aligned Grid

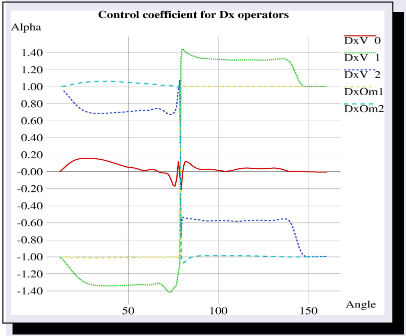


Conventional BC, Rotated (30°) Grid

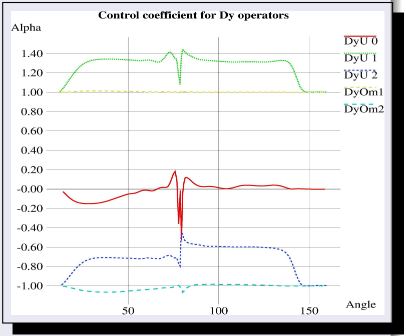


Optimized BC, Rotated (30°) Grid

Single gyre, α_1, α_2



Derivatives in x



Derivatives in y

$$\left.\frac{\partial v}{\partial x}\right|_{\omega_\circ} = \frac{\alpha_1 v_{-1/2} + \alpha_2 v_{1/2}}{h}$$

$$\left.\frac{\partial u}{\partial y}\right|_{\omega_\circ} = \frac{\alpha_1 u_{-1/2} + \alpha_2 u_{1/2}}{h}$$

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$$\left.\frac{\partial \omega}{\partial x}\right|_{v} = \frac{\alpha_1 \omega_0 + \alpha_2 \omega_1}{h}$$

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$$\alpha_1 = 0, \quad \alpha_2 = -0.1$$

$$\alpha_1 = -1.4, \quad \alpha_2 = 0.6$$

$$\alpha_1 = -1.01, \quad \alpha_2 = 1.01$$

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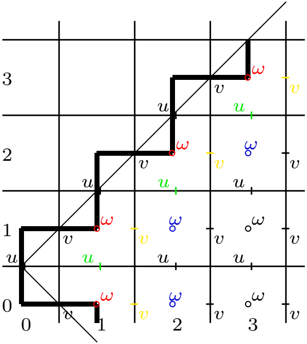

Рис.: 45° rotated grid

$$\omega_\circ = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - 0.8\frac{u+v}{h}$$

$$\omega_\circ = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{u+v}{h}$$

$$= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{\vec{V}.\vec{\tau}}{h/\sqrt{2}}$$

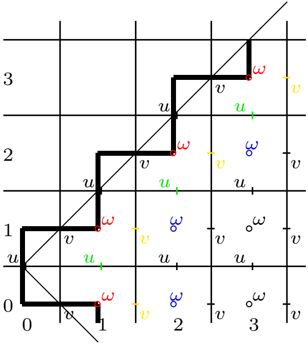

Рис.: 45° rotated grid

"Optimal"configuration

$$
\begin{aligned}
\omega_\circ &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{u+v}{h} \\
&= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{\vec{V}.\vec{\tau}}{h/\sqrt{2}}
\end{aligned}
$$

- Free-slip condition on a curvilinear boundary $\omega|_{bnd} = \frac{\vec{V}.\vec{\tau}}{R}$
- Optimal boundary is curvilinear with $R = -h/\sqrt{2}$

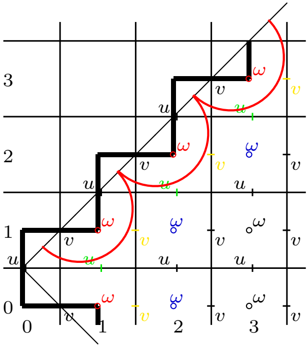

Рис.: 45° rotated grid

$$\omega_\circ = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{u+v}{h} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{\vec{V}.\vec{\tau}}{h/\sqrt{2}}$$

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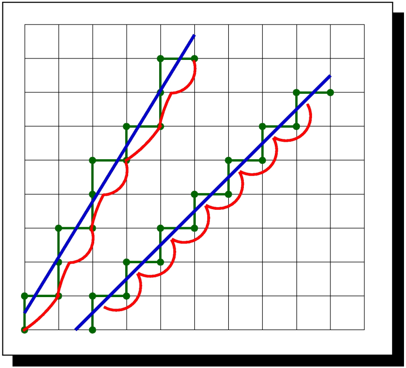
Variable R_{30° : $-h \leq R_{30^\circ} \leq 5h$.



"Optimal"configuration

$$\omega_\circ = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{u+v}{h}$$
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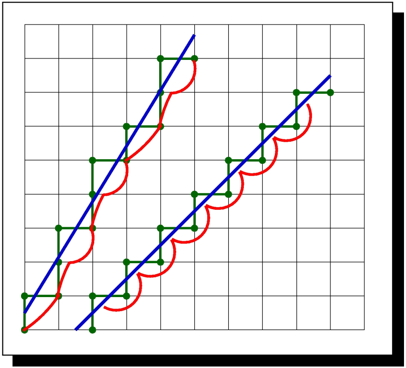
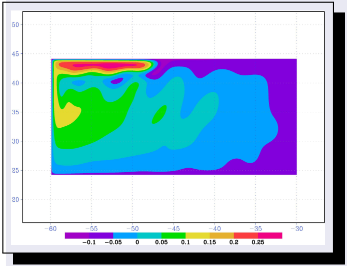
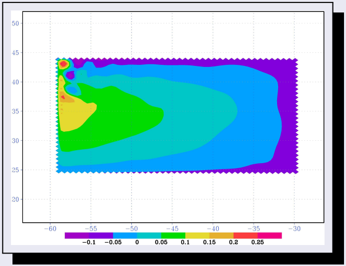


The Radius depends on the grid resolution h.
Does this curvilinear boundary remains optimal on different resolutions?

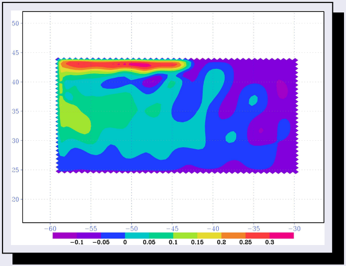
Reference, Optimal and Conventional BC 800 days later



Reference SSH

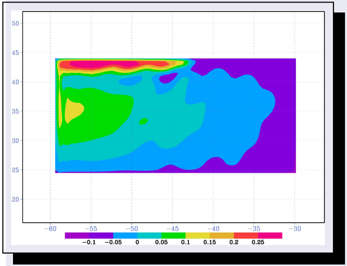


Rotated grid conventional BC SSH

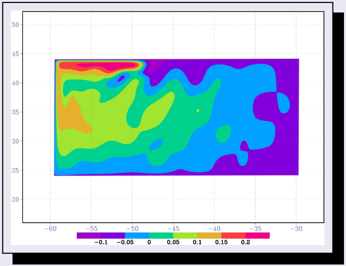


Rotated grid constant $R = -h/\sqrt{2}$ BC SSH

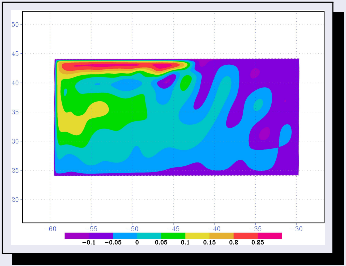
Reference, Optimal and Conventional BC 800 days later



Reference SSH



Rotated grid conventional BC SSH

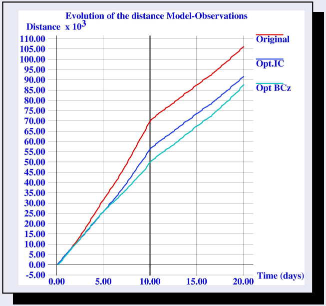


Rotated grid, constant $R = -h/\sqrt{2}$ BC SSH

Layout:

- NEMO, Global ocean model, 2° resolution, 31 layer;
- ECMWF data issued from Jason-1 and Envisat altimetric missions and ENACT/ENSEMBLES data banque;
- Data Assimilation during 10 days interval;
- Analysis of the distance "model–observations"on 1 month interval

Distance "model–observations"



ORCA-2 Model

Layout:

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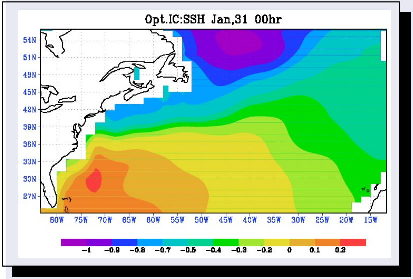
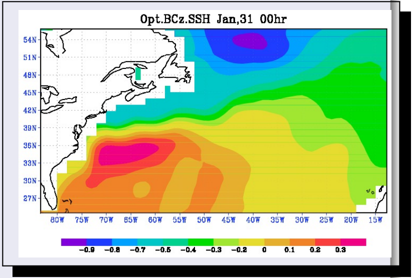
- Data Assimilation during 10 days interval;
- Analysis of the distance "model–observations"on 1 month interval

SSH, North Atlantic, January, 31, 2006.



Optimal Initial Conditions.



Optimal BC at the bottom.

Conclusion

Boundary Conditions influence is important

- Optimal BCs allows **to correct errors** committed by the discretization
- The model **is closer** to the reference one with optimal BC
- Data assimilation allows to get an **optimal position and form** of the boundary

BUT

As well as for any adjoint parameter estimation

- The control may violate the model physics;
- The physical meaning of the optimal boundary is difficult to understand;
- The set of α is not unique;
- The problem of identifiability is not addressed yet;
- The problem of stability is not even posed.

Consequently:

It is not a parameter estimation study, but

- a way to compensate model errors
- showing the most influent parameter.

Another result of the Russian-French cooperation

2001

2015